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THE MATHEMATICS TEACHER

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A RIDDLE FROM ARCHIMEDES

BY GEORGE W. EVANS

Houston, Texas

What we are here concerned with is the question how Archimedes got his two remarkably accurate approximations to $\sqrt{3}$. There have been many guesses at the answer. They fall into two classes, the first based upon Euclid's formula for approximating to $\sqrt{2}$, and the second based partly upon Heron's description of the process he used to find $\sqrt{720}$. Of all these, none is entirely convincing.

Archimedes was proving that the ratio of a circle to its diameter is less than $3\frac{1}{7}$ and greater than $3\frac{10}{71}$. His proof depended, as in the books of our boyhood, on doubling the number of sides in circumscribed and inscribed polygons. He computed, not the length of the perimeter in his resulting polygon, but its ratio to the diameter. He made use of the ratios that we call the cosecant and the cotangent of half the central angle, and had a routine, where we would use a formula, for finding the new values of these functions as he went on halving the angle. As he started with the hexagon, he needed approximations for $\sqrt{3}$; and since he was proving that π was greater than one number and less than another, he needed to show some circumscribed perimeter not greater than the upper limit and some inscribed perimeter not less than the lower limit. He took, then, to use with the inscribed polygon, $1351/780 > \sqrt{3}$, and $265/153$ to use with the inscribed polygon.[1]

The First Kind of Guess.—If we now go back to Euclid's *Elements,* we find in the tenth proposition of the second Book a proof of the identity

$$(2a + c)^2 + c^2 = 2a^2 + 2(a + c)^2.$$

[1] $1351/780 = 1.732051+$; $265/153 = 1.732026+$; $\sqrt{3} = 1.7320508+$.

This is one of the propositions, occurring frequently in Euclid, where he used his clumsy geometrical algebra to prove a theorem about numbers. It is the same as the rule [2] for finding successive approximations, in integers, for the relative lengths of the side and diagonal of a square; these were later called "side and diagonal numbers." The tradition about them goes back to the time of Pythagoras.

If a represents the length of the side of a square and c the length of its diagonal, it should, of course, be true that $c^2 = 2a^2$; but when c and a are only approximate there will be an error of area, that is, a difference between the square on c and double the square on a. The rule enables us to get successively larger values for c and for a, *without changing the value of this error of area.* We take $(2a + c)$ for a new value of c, and $(a + c)$ for a new value of a; and the theorem shows that

$$(2a + c)^2 - 2(a + c)^2 = 2a^2 - c^2.$$

The obvious purpose of the formula is to find ratios of larger integers representing the relative lengths of c and a, without changing the magnitude of the error of area. Thus, starting with $c = 3$ and $a = 2$, we get by four successive substitutions the fraction 99/70, or 1.41429 —, while $\sqrt{2} = 1.41421 +$.

As the notation of algebra improved, it became possible to find formulas somewhat like this for approximating to the square root of any integer. It also became possible to approximate to these square roots in other and more convenient ways, so that this method became for actual computation of little use, and was for its original purpose almost entirely forgotten. It led, however, to the discussion of certain indeterminate equations, in which some of the most famous mathematicians of several centuries were involved. The discussion has continued to our own times, and has a voluminous and very important literature.[3]

A little elementary algebra will show that if $p^2 - nq^2 = -1$, then $(pc + nqa)^2 - n(qc + pa)^2 \equiv -(c^2 - na^2)$; and if $p^2 - nq^2 = +1$, then we find that $(pc + nqa)^2 - n(qc + pa)^2 \equiv + (c^2 - na^2)$. We shall refer to these identical equations as (1) and (2) respectively.

[2] Heath's *Euclid*, Vol. I, p. 398.

[3] See L. E. Dickson, *History of the Theory of Numbers*, Vol. II, Chap. XII; also E. E. Whitford, *The Pell Equation*, Columbia University Press, 1912.

Equations of the first type, like (1), can be found for an infinite number of values of n, but not for all. With these equations the error of area remains constant in amount, but regularly changes in sign. Equations of the second type, like (2), can be found for any integral value of n that is not itself a perfect square. You have only to find one pair of values for p and q that will satisfy $p^2 - nq^2 = +1$. In the time of Archimedes this could have been done only by trial; an unfailing algorithm for it was not known to Europeans until the seventeenth century, when a brilliant Irishman named Brouncker, the first president of the Royal Society, furnished it to Wallis. The Arabs had hit on it earlier, but had not been found out.

Testing Out the Guess.—Now $2^2 - 3(1)^2 = +1$; so that we get, by substituting $n = 3$, and $p = 2$, and $q = 1$ in equation (2), the approximation formula

$$(2c + 3a)^2 - 3(c + 2a)^2 = c^2 - 3a^2,$$

and from this, by successive substitution, comes the following table:

	$c=2$	$a=1$	$c=7$	$a=4$	$c=26$	$a=15$	$c=97$	$a=56$	$c=362$	$a=209$
$2c$	4		14		52		194		724	
$3a$	3		12		45		168		627	
c		2		7		26		97		362
$2a$		2		8		30		112		418
	7	4	26	15	97	56	362	209	1351	780

For every one of these pairs of numbers the equation $c^2 - 3a^2 = 1$ is true, so that $c^2 > 3a^2$, $c^2/a^2 > 3$, $c/a > \sqrt{3}$. The ratio 1351/780 is the value Archimedes took for $\sqrt{3}$ in discussing the inscribed polygon.

For the circumscribed polygon he needed values of c and a such that $c/a < \sqrt{3}$, $c^2/a^2 < 3$, $c^2 < 3a^2$, and $c^2 - 3a^2 =$ some negative number.[4] There is no solution for the equation $c^2 - 3a^2 = -1$, but we have $5^2 - 3(3^2) = -2$, and if we start with $c = 5$ and $a = 3$ in our equation for $c^2 - 3a^2$, we get the following table of successive values for c and a:

[4] Negative numbers had not yet been invented. Archimedes would have said he needed values of c and a such that $3c^2 - a^2 =$ some small number.

	$c=5$	$a=3$	$c=19$	$a=11$	$c=71$	$a=41$	$c=265$	$a=153$	
$2c$	10		38		142		530		
$3a$	9		33		123		459		
c		5		19		71		265	
$2a$		6		22		82		306	
	19	11	71	41	265	153	989	571	

For every one of these pairs of numbers the ratio $c:a < \sqrt{3}$, which is just what Archimedes wanted. We should probably have selected $989:571$ instead of $265:153$, which is the ratio that he used; or we would have taken from the other table $362:209$ rather than the larger numbers that he took. We do not know whether Archimedes was really able to make tables like these, and was thus able to choose. In any case he might have persisted in his choice, for the following reasons. When he was computing with the larger numbers he was able twice to reduce his values by cancellation. Except where he found cancellation possible his denominators were constant, and 780 has several small factors; on the other hand 571 is a prime number, and the ratio $989:571$ would have led to useless labor. With the notation that the Greeks had, such labor was somewhat more irksome than with us.

However, Archimedes did not have our algebraic notation, nor was he likely to have in mind our generalizations or such a series of approximations as we have here shown. He evidently did have in mind two different methods for square root. One was for large numbers, where, the square root being between two consecutive integers easily ascertainable, the fractional part would need only a rough approximation. This fractional part he probably found more or less as we do. The other method, which the world has long been searching for, he used for $\sqrt{3}$.

The Second Kind of Guess.—Whatever method he used for this last purpose, it probably involved no new or unusual theorems, for it passed without comment either by Archimedes or by his successors or commentators. Of the conjectures made in regard to it in more modern times, the most favored is based partly on the procedure described by Heron of Alexandria, who lived somewhere near the beginning of the Christian era. As told by Heron himself, this is his way of finding the square root of 720.

"Since 720 has not its side[5] rational, we can obtain its side within a very small difference as follows. Since the next succeeding square number is 729, which has 27 for its side, divide 720 by 27. This gives 26 and 2/3. Add 27 to this, making 53 2/3, and take half of this or 26 5/6. Indeed, if we multiply 26 5/6 by itself the product is 720 1/36, so that the difference is 1/36. If we desire to make the difference still smaller than 1/36, we shall take 720 1/36 instead of 729, and by proceeding in the same way we shall find that the resulting difference is much less than 1/36."[6]

This method, by the way, has recently been revived in the United States for pedagogical reasons, and the prestige of antiquity has been added to it quite unexpectedly; for Heron's use of it was not known until the manuscript of his "Metrica" was discovered at Constantinople in 1896, or at least until a fragment containing this rule was discovered in Paris in 1894. Publication did not come until some years later, so that the suggestion came from tradition—or suspicion—rather than from history.

There is nothing to show that this method was used by Heron for numbers smaller than 720. Some such results could have been obtained in this way, but some others could not; and all could have been obtained otherwise. For example, he gives $\sqrt{63}$ as 7 15/16, which could have been found by taking 8 as a trial root; but $\sqrt{135}$, which in the same context is given as 11 13/21, could not have been so found except by taking 11 2/3 as the trial root. Both these results, however, are involved with the solutions of the equation $p^2 - nq^2 = \pm 1$, for which the Greeks knew some dandy tricks, though they certainly had no complete theory.

Handy Anachronisms.—Whatever Heron himself may have done with his rule, it will not relieve our perplexities about the values Archimedes took for $\sqrt{3}$. If we allow a fraction for the trial root, we could start with 1 2/3 and obtain 26/15 à la Heron; and from that in the same way we could get 1351/780, which is all right; or it would not be unreasonable to suppose 26/15 already known by trial or otherwise, and handed down by tradition. But Heron's excellent rule will not reach 265/153. To get this we have to introduce a new rule, first found in the writ-

[5] He means its square root.

[6] Heath, *History of Greek Mathematics*, Vol. II, p. 324.

ings of an Arabian algebraist of the eleventh century.[7] So far as any evidence goes, it really seems quite unlikely to have been at the command of Archimedes or even of Heron.

One More Guess.—There is another explanation, which very likely may have already been made, but in my scattered reading I have not happened to find it. It is suggested by the fact, noted by Whitford, that if we have a pair of numbers p_1 and q_1 satisfying the equation $p^2 - nq^2 = 1$, a series of new and larger values of p and q may be found in the successive powers of $p_1 + q_1\sqrt{n}$; and if p_1, q_1 and x_1, y_1 are two such pairs of numbers, then $(p_1 + q_1\sqrt{n})(x_1 + y_1\sqrt{n})$ will give another pair.

All the commentators seem willing to admit that the equation $26^2 - 3(15^2) = 1$ was very likely known to Archimedes and the rest, perhaps having been obtained by trial. This equation can be written $(26 + 15\sqrt{3})(26 - 15\sqrt{3}) = 1$. From this, by squaring, comes $(1351 + 780\sqrt{3})(1351 - 780\sqrt{3}) = 1$. Then the equation $1351^2 - 3(780^2) = 1$ gives the approximation that was used for the inscribed polygons. Before considering the other value, let us see whether these operations were within the scope of mathematical learning in the lifetime of Archimedes.

The resources of computers then, as now, depended on algebra; but their algebra was of the type exemplified in Euclid, not merely in Book II, but for example in X, 97, which is stated as follows:

> *The square on an apotome applied to a rational straight line produces as breadth a first apotome.*

The theorem is made obscure by technicalities, but its meaning is that if an expression like $26 - 15\sqrt{3}$ is squared and divided by a rational number, the quotient is an expression of the same form. Taken in connection with II, 7, which proves $a^2 + b^2 = 2ab + (a - b)^2$, it serves to make clear that the equation $(26 - 15\sqrt{3})^2 = 676 + 675 - 780\sqrt{3}$ would be evident to the adepts after Euclid.

In a similar way X, 60, is an extension to irrationals ("binomials" this time) of II, 4, which proves $(a + b)^2 = a^2 + b^2 + 2ab$. Together, they warrant the equation $(26 + 15\sqrt{3})^2 = 1351 + 780\sqrt{3}$.

Finally, 112, 113, 114, of Book X, prove that if a rational

[7] $\sqrt{a^2 + b} > a + (b/[2a + 1])$; see Heath, *Hist. Greek Math.*, II, p. 51.

number is divided by $x - \sqrt{y}$, the quotient is proportional to $x + \sqrt{y}$; if it is divided by $x + \sqrt{y}$, the quotient is proportional to $x - \sqrt{y}$; and the product of $x + \sqrt{y}$ by $x - \sqrt{y}$ is a rational number. There is also, in II, 5, the proof of the equation $(a + b)(a - b) = a^2 - b^2$. Now it is true that the algebra of that time was nowhere near so convenient as ours, but when we consider the extraordinary skill with which this halting and vague instrument of investigation was handled, we cannot doubt the acceptance, even in Euclid's time, of the change from $26^2 - 3(15^2) = 1$ to $(26 + 15\sqrt{3})(26 - 15\sqrt{3}) = 1$; or from $(1351 + 780\sqrt{3}) \times (1351 - 780\sqrt{3}) = 1$ to $1351^2 - 3(780^2) = 1$.

The Greeks represented numbers by straight lines. The line for $\sqrt{3}$ would be the side of an equilateral triangle inscribed in a unit circle. A multiple of that line, and expressions like $26 + 15\sqrt{3}$, could easily take their place in the algebraic system. Products of any numbers represented by lines were rectangles, and lines to represent such products could be found by "applying" the rectangles to the unit straight line, that is, by constructing an equivalent rectangle on that line as base. Division was performed by means of this same "application of areas." But, "inasmuch as one straight line looks like another, the Greeks did not get the same clear view of what they denoted as our system of symbols assures to us."[8] In consequence, we see that in all the three cases mentioned above, Euclid appears to have felt the need of proving the results of Book II applicable to the irrational magnitudes of Book X. There was really no such need, since the question of commensurability is not involved in Book II at all.

The other approximation, 265/153, which was the lower limit, could have been obtained by writing $3(3^2) - 5^2 = 2$ in the form $(3\sqrt{3} + 5)(3\sqrt{3} - 5) = 2$, and multiplying this by $(26 + 15\sqrt{3})(26 - 15\sqrt{3}) = 1$, so as to obtain as product $(153\sqrt{3} + 265)(153\sqrt{3} - 265) = 2$, which shows that $3(153)^2 - 265^2 = 2$. The only difficulty here is to understand clearly how these multiplications could have been dealt with by the methods current in the third century B.C.

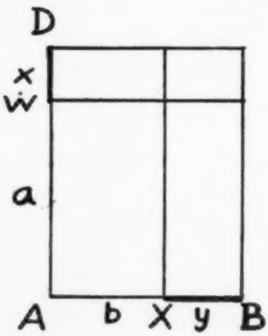
Expressed in our algebra, there are two theorems involved. They are in the style of Book II, as follows:

[8] Zeuthen, quoted by Heath in his *Euclid*, Vol. III, p. 5.

(1) $(a+x)(b+y)=ab+bx+ay+xy$; and (2) $(a-x) \times (b-y)=ab+xy-bx-ay$.

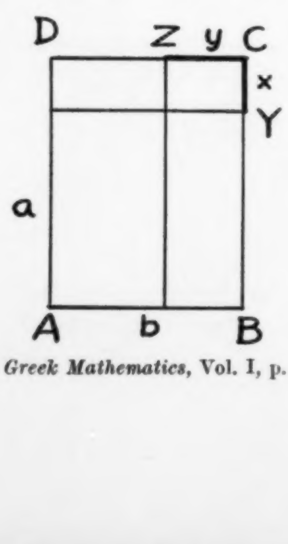
For each of these identities we need the following meanings for the letters: $a=26$, $b=3\sqrt{3}$, $x=15\sqrt{3}$, $y=5$.

For (1) we take, in Fig. 1, $AW=a$, $AX=b$, $WD=x$, $XB=y$. Then the entire rectangle $(a+x)(b+y)$ is equal to the



"rectangles about the diagonal," namely ab and xy, together with the "complements" ay and bx. Obvious though it is, its statement in words would be cumbrous. It is an unlovely thing, then, to put in a book, though the ancient multiplications of mixed numbers[9] show that its import was well known; for such use a and b in Fig. 1 would represent the integral parts of the mixed numbers, x and y the fractions.

For (2) we use in Fig. 2 the same diagram with a different allotment of parts. Here $AD=a$, $AB=b$, $YC=x$, and ZC



[9] Heath, *History of Greek Mathematics*, Vol. I, p. 58.

$=y$. Then, by analogy with Euclid, II, 7, where it is proved that $a^2+x^2=2ax+(a-x)^2$, we note that $ab+xy$ (the entire rectangle and one of the rectangles about the diagonal) is equal to the two rectangles ay and bx, together with the other rectangle about the diagonal, $(a-x)(b-y)$; that is,

$$(a-x)(b-y)+ay+bx=ab+xy.$$

Our result has the genuine Euclidean avoidance of the negative sign. Euclid would no more think of taking a line from a smaller line than he would think of taking a pail of water out of a cup. His definitions of binomial and apotome are in substance as follows:

> *Of two straight lines, incommensurable in length but commensurable in square, the sum is a binomial and the difference an apotome.*

The binomial, as well as the apotome, is shown to be incommensurable with either of the two original lines, even in square. In the case of the apotome, the line subtracted is called the "excess," and the other, the larger line, is called the "whole."

With this punctilio in mind, we can try our hand in phrasing for Euclid some of the theorems he was so heedless as to omit. To deal with the approximation 265/153 for $\sqrt{3}$, we need, as we have seen, two theorems in continuation of Book II. The second one, referring to the product of two different apotomes, would perhaps have read thus:

> *If two given lines be unequal, and if they be divided each into two unequal parts, the rectangles contained by the given lines and by the smaller parts, taken together, are equal to the rectangle contained by the larger parts and the two rectangles contained each by the larger part of one and the smaller part of the other.*

We can hardly blame him for leaving it out. Indeed, if he had included it in his Book II, he would have probably felt the need of adding to Book X, and proving by his unwieldy proportions, the following theorem also:

> *If there be given two apotomes, and if the excess of each be commensurable with the whole of the other, then the rectangle contained by the apotomes, if applied to a rational straight line, produces as breadth a second apotome.*

J. L. Heiberg, the latest and best of the authorities on Euclid's text, could not see any use for the theorems 112, 113, 114, of Book X, which we have referred to above, and on that account thought that they were probably spurious; it is easy to imagine with what scorn he would have rejected this last concoction of ours.

WILLIAM JAMES AND HENRI POINCARÉ [1]

By PROFESSOR M. H. INGRAHAM

Brown University, Providence, Rhode Island

Mr. Chairman, Ladies and Gentlemen: A good many times today I have heard it said that we should emphasize in class those things that would interest the man in the street. I suppose that this is good doctrine. But if it is sound, it seems to me that our most important task is to find the right man in the street. Some years ago I succeeded in doing this to a marked degree here in Worcester. Between my sophomore and junior years in college I spent some time tramping in the White Mountains and on the way took the opportunity to see a little of New England. I had a very pleasant trolley trip starting from Hartford, including Springfield, Amherst, Northampton and ending here in Worcester. I had never been off the train in Worcester before and had only a few hours to stay. I knew nothing of the city except its population and the hours of a few departing trains. What should I see? What should I do? I went to the man in the street, the first policeman that I saw, and told him my plight. To my surprise he directed me to the Art Gallery. It was an unexpected answer but a very good one, for it is truly a gallery of which to be proud.

This is not the place, therefore, to feel apologetic in choosing to speak on certain aspects of the thoughts of two scientists, when these supposedly matter-of-fact people, a mathematician and a psychologist, policemen of the mind charged with the establishment of order among the crowding, jostling thoughts of the period, in answer to the question what one should really look for in science, answer together "look for beauty, esthetic values." Surely an audience in Worcester will be sympathetic with such a thesis and tolerant of the stumbling purveyor of it.

If we were to ask who were the truly great men flourishing in 1700, the perspective that history has given would probably

[1] Delivered at the Winter Meeting of the Association of Teachers of Mathematics in New England, March 5, 1927.

make our choice of some value. Among these, far ahead of soldiers like Eugene and Marlborough, of potentates and pontiffs, of even Louis XIV, would have to come men like Newton and John Locke, the leading mathematician, astronomer and physicist of his age and a leader of philosophy and psychology. If we were asked to name the great of 1900, what a welter of confusion we would be in! Clemenceau? Roosevelt? Foch? No end of argument! However, it would be surprising if the future did not rank among the highest the greatest mathematician and mathematical physicist and astronomer of his age, Jules Henri Poincaré—and a leading philosopher and psychologist, that rugged, good-humored, pleasantly combative personality, William James. Surely for us as teachers of mathematics, psychologists and mathematicians, especially if they are philosophers besides, must be of interest.

William James was born in New York in 1842 of remarkable parents. His father had a speculative turn of mind accompanied by a gift of expression, that descended on both his sons, William and Henry. James' youthful surroundings were intellectual. The household was permeated by a faith in a few essentials, that was totally unafraid to face and share in most alarming scepticisms. After a desultory schooling or lack of schooling at home and abroad he entered Harvard and came under the influence of such men as Elliott and Agassiz. He proceeded to the medical school from which he graduated in 1869. From 1872 to 1881 he was instructor and assistant professor in physiology and from 1881 to 1907 professor of philosophy. He died in 1910.

Poincaré also belonged to a family of distinction. His father was a successful doctor in Nancy, where Henri was born in 1854. He was the cousin of the present premier of France, and his sister married Emile Boutroux, the French philosopher, and moved with distinction in the most intellectual of circles. From boyhood Poincaré's life was devoted to scholarship. After nearly failing to be allowed to enter the Lycee because of a poor examination in mathematics and physics he made a good record and proceeded to the Ecole Polytechnique where he was a student from 1873–1875. After experience as an engineer and some teaching he was appointed to a chair in Mathematical Physics

and Probabilities at the Sorbonne in 1886, where were most of his duties till his death in 1912.

I give this brief survey of the lives of these two men chiefly to point out that in considering a few of their philosophical conclusions we must remember that different as New York was from Alsace, different as Cambridge is from Paris, different as psychology is from mathematics and physics and, most of all, different as the temperaments of these two men were, still both approached philosophy with the point of view of men experienced in scientific thought and research, the one an anatomist, physiologist and psychologist, the other a physicist, astronomer and, most of all, a mathematician.

Although, in what I am about to say, I will mention some differences between the thoughts of these two men, and the differences are everywhere, I shall mainly stress some points of agreement. Even on these points I presume they would be scandalized to find themselves mentioned together. But certainly both agreed that one of the duties of the mind was to correlate and bring into agreement diverse phenomena and whatever they were to themselves they are merely diverse phenomena to us.

There are four important theses that, in one form or another, they agreed on as a result of the analysis of their own thought processes in scientific work. First, there is no use of our talking about any hypothetical absolute reality outside of human experiences and thought processes. But after this statement, sounding like a declaration of independence from reality, comes the second statement that systems of thought not stimulated by our elemental experiences, experiences of reality, are almost sure to be sterile. Thirdly, the most important things to be decided as to a hypothesis are not the a priori reasons for its choice, but what its results are. Who was it that said "I won't grant that until you tell me what you are going to prove with it"? It could have been either James or Poincaré. Finally, and most of all, they both recognized the rôle of æsthetic feeling in science and philosophy, both as a guide to new results and as a justification of the worthwhileness of systems of knowledge.

For ages philosophers have tried to build up a world they called real, different from the physical things we sense or even physical theories we construct. A world where goodness exists

per se, not as concrete examples of honest words and kind acts but as a disembodied. They talked of twoness and threeness as existing per se not in concrete examples, not even in man's thought, but as a disembodied. They would even talk of chair and fruit as something beyond all our experiences of Windsors or bananas as something preexisting, not concrete but as a disembodied. By this time you are all probably thinking that my grammar has gone totally awry, that I have forgotten to find the noun for disembodied to modify. Not at all. I have not forgotten. I have failed. Any noun would have far too much concreteness to give any correct impression of the other-than-worldliness and other-worldliness of this conception.

How James arises to smite them! In speaking of things not in our mental experience his dictum was clear. "In my opinion, we should be wise not to consider any thing or action of that nature, and to restrict our universe of philosophic discourse to what is experienced or, at least, experienceable." [2]

James forever insists that what we experience in our mental processes affords the most important realm for our consideration, that by postulating this other world of abstractions and calling it by some strange perversion, the real world, we are taking our problem so far away from anything of interest to us that it is useless. But such a statement gives no idea of the joyousness with which he enters the conflict. He loves the battle so much that he loves the enemy because they are willing to be the enemy. It is possible to read James when he is expounding his conception of the truth riding forward like Sir Galahad in quest of the Grail, but give him an opponent and the coldness of Sir Galahad, whose "strength is as the strength of ten because his heart is pure," vanishes. The warm rich fighting blood of Lancelot flows in his veins, a rush, "one touch of that skilled spear" and the other knight is sent with joyful good nature to report to King Arthur. One more abstractionist (such is their terrifying name) is brought to earth.

Let me illustrate these two styles of James. Here is one sentence of James, expounding, "The natural result of such a world-picture has been the efforts of rationalism to correct its incoherence by the addition of transexperiential agents of unification, substances, intellectual categories and powers, or selves;

[2] *Essays in Radical Empiricism*, page 243.

whereas, if empiricism had only been radical and taken everything that comes without disfavor, conjunction as well as separation, each at its face value, the result would have called for no such artificial correction."[3] I hope you understand.

But listen to these good-natured but adroit thrusts at his opponents.

In referring to Mr. Joseph: "He has seriously tried to comprehend what the pragmatic movement may intelligibly mean; and if he has failed, it is the fault neither of his patience nor of his sincerity, but rather of stubborn tricks of thought which he could not easily get rid of."[4]

Or, again, this parry and thrust, "Mr. Russell himself is far too witty and athletic a ratiocinator simply to repeat the slander dogmatically. Being nothing if not mathematical and logical, he must . . . convict us not so much of error as of absurdity. I have sincerely tried to follow the windings of his mind in this procedure, but for the life of me I can only see in it another example of what I have called vicious abstractionism."[5]

But enough of James' style. What does the quiet logical mathematician Poincaré feel? One, perhaps, would expect that he would have more sympathy with the less empirical attitude. His style is far less vivid but far more clear. James is always vivid, if not clear, for assuredly that is possible. Consider the feelings of a man the moment a shovel full of snow falls from a roof upon him. The sensations are surely vivid, but clarity is hardly their chief characteristic. But, as I have said, Poincaré is always lucid, and here he strongly agrees with James.

Poincaré states the case more succinctly, if less often, than James. "Beyond doubt, the reality completely independent of a mind which conceives it, sees or feels it, is an impossibility. A world as exterior as that, even if it existed, would for us be forever inaccessible. But what we call objective reality is, in the last analysis, what is common to many thinking beings and could be common to all."[6]

But still no one could think of a mathematician who was not devoted to abstract systems, and James recognizes the value of

[3] *Essays in Radical Empiricism*, page 43.
[4] *Essays in Radical Empiricism*, page 244.
[5] *The Meaning of Truth*, page 276.
[6] *The Foundations of Science* (Value of Science), page 209.

the broad generalizations of science even, perhaps a little grudgingly, of mathematics. The point is, we recognize these systems as inventions of our mind, not as something true outside of human experience. What guide should we have for such abstract systems? Why do we teach geometry? Why do we talk about a point? No one ever saw one, felt one, heard one, tasted one, or smelt one. No one ever even performed a physical experiment on one. Still worse are those strange collections of points, called lines, or planes, or triangles, or circles. All of these are objects utterly devoid of physical existence. And yet we insist that no one shall leave our schools without being exposed to a vast system of abstract statements about these things, called Plane Geometry. Is it because of mental drill? Try chess and have the fun of competition thrown in to boot. Is it the love of logic? The medieval theologians arguing about the number of angels that could dance on the point of a needle showed, probably, equal dialectical skill. Is it love of abstraction? It cannot compare with some of these terrible philosophical theories we have just condemned at such length. If you say it is because of the beauty of the subject, you are indeed among those who have received the mathematical blessing. But even then, you must say in what this beauty consists. No! Even those of us who most love pure mathematics as such must admit that much of the importance of the geometry we teach is due to the fact that abstract, unreal if you wish, as it is, it was inspired by the everyday experiences of the world. Points grew from dots, small bits of actual matter. Lines from taut strings, beams of light, edges of knives, etc. Planes came from calm seas, circles from horses walking around a tread-mill, and all these theorems that we so laboriously prove about them seem also to be the counterpart of all sorts of facts we actually see. Have any of you ever marked a tennis court for the first time? Your guide strings are not straight lines, the court unfortunately is usually a very poor approximation of a plane, and yet after laying off 78 feet one way, and 36 another, you, as a check, measure the diagonal to see if it is a trifle under 86 feet, and then play in calm faith that Pythagoras has guaranteed the accuracy of your work. If it is not the physical reality of geometry, it is at least its contacts with physical reality that gives to it much of its value. Trigonometry did not make surveying possible, but

surveying made trigonometry inevitable. This again is in the main line of thought of both of our philosophers, though they differ in many ways. James forever ties things down to everyday experiences. He can scarcely be said to bow the reverent knee to mathematics. He berates Russell with being too mathematical, and yet he builds his own system of thought and is too good a scientist not to recognize the value of abstraction, tacking on always some such statement as the following which is clear as to the one use of abstraction that James considers to be the only. "To be helped to anticipate consequences is always a gain, and such being the help that abstract concepts give us, it is obvious that their use is fulfilled only when we get back again into concrete particulars by their means, bearing the consequences in our minds, and enriching our notion of the original objects therewithal." [7]

Poincaré has a far higher concept of the rôle of abstraction and to my mind a truer one, but still a very real sense of the concrete. Here is one statement. "It is only through science and art that civilization is of value. Some have wondered at the formula: Science for its own sake; and yet it is as good as life for its own sake, if life is only misery; and even as happiness for its own sake, if we do not believe that all pleasures are of the same quality, if we do not wish to admit that the goal of civilization is to furnish alcohol to people who love to drink." [8]

But this follows upon statements of gratitude to physics. "In the first place the physicist sets us problems whose solution he expects of us. But in proposing them to us, he has largely paid us in advance for the service we shall render him if we solve them. If I may be allowed to continue my comparison with the fine arts, the pure mathematician who should forget the existence of the exterior world would be like a painter who knew how to harmoniously combine colors and forms, but who lacked models. His creative power would soon be exhausted." [9]

The third point is still more vital for us. There is little danger of any of us, as mathematicians, refusing to talk abstractly and use generalized methods. The tremendous power of algebra and geometry are evident to us. On the other hand,

[7] *The Meaning of Truth*, page 246.

[8] *The Foundations of Science* (Value of Science), page 355.

[9] *The Foundations of Science* (Value of Science), page 284.

students, forever demanding concreteness, will keep us anchored to reality but all of us have time and again asked ourselves, how did any one ever make up the axioms of geometry or formulate the fundamental laws of algebra? What are the ear marks of a sensible hypothesis? Shall we limit ourselves to so-called "self-evident truths"? What can we say if to someone they are not self-evident? Here is the answer, not quite obvious at first, perhaps even startling to some, but which seems to grow on one. We should not try to justify hypotheses so much by showing that they are intrinsically plausible as by showing that they lead to results that work. "By their fruits ye shall know them." Suppose we agree that $a^m a^n = a^{m+n}$ when m and n are integers. Is that any great reason for thinking that the same rule holds when they are not? Yet we agree to use this rule in all cases. You ask me why, and I must answer because it works. It simplifies many calculations and under certain circumstances it is the only hypothesis that will do this.

James has two books, "Pragmatism" and "The Meaning of Truth," in which his main thesis may be summarized as: A true statement is one that works. It takes a lot of explaining on his part for one to fully understand a statement such as: "The true, to put it very briefly, is only the expedient in the way of our thinking, just as the right is only the expedient in the way of our behaving."[10] On this point James' enthusiasm is so great that it sometimes leaves the more slow-minded of us in doubt. Yet the underlying thought is correct.

We will never know that our most useful statements about nature are true in any absolute sense though it is quite likely that we may find they are not. We will value them, however, in the degree in which they "summarize old facts" and "lead to new ones."[11]

It is just in this connection James makes the only reference to the philosophy of Poincaré that I have run across, and that only in passing.

Poincaré, never like James, tries to define truth in terms of expediency of thought or convenience. To him it seems to have had an absolute meaning, but one would have to agree that between them there is little essential difference. James talks of

[10] *The Meaning of Truth*, Preface, page vii.

[11] *Pragmatism*, page 57.

truth, but emphasizes the workability of statements. Poincaré chooses to talk of hypotheses, but it is still the same use of summarizing old facts and leading to new that he emphasizes. He is rich in the number of examples that he brings forth of hypotheses (now discarded) that have still played an enormous rôle in promoting further research and leading to new facts. If relativity is to be supreme for a while, could it have arisen without the work of Newton? The present theories of evolution are far from Darwin's, yet how greatly they have been affected by his work. Probably none of the present scientific theories of nature are final, yet how much science does to help us actually control nature. If we were to try, in any absolute sense, to prove to an intelligent sceptic the truth of any given scientific theory picked at random, I would bet on the sceptic. But we will not need to stop believing in the value of theories—their value is proved by their workableness—they have a pragmatic sanction.

However, if the reading of the writings of these two scientists not only brought to me some interest and, in many points, assent to their logical arguments, but also left me bubbling over inside with a real sense of inspiration and elation, it was not from their analysis of what science is and how it works, but it was from the fact that throughout all their work is shown the spirit of men who dwell with the beautiful. It is their sense of the æsthetic value of science, and also of the value of the sense of the beautiful to science that impresses one most. And yet, it is at this point of strongest agreement that is most strikingly illustrated their difference of temperament, for in their judgment of what is beautiful is the greatest antithesis.

Poincaré is dominated by the sense of law, of harmony, of clarity, of unity. James loves diversity, the unexpected, the richness and variety of detail, the half-felt sensations.

In a sense, both mathematics and what we call psychology are separate branches of the study of how the human mind works, but mathematics studies how it works according to formulated law, according to necessity, it is the branch of psychology with a certain universality of application, far different than the psychology that tries to explain the vagaries and wanderings of the human mind in all the ins and outs of our daily experiences. It was inevitable that Poincaré would be attracted to the one—

James to the other. The same tendencies are felt in every art. You have your Raphael clear cut, balanced, well ordered—and you have your Corot or Whistler vague and suggestive. In poetry one has a Pope or Dryden on the one hand, a Shelley, Walt Whitman, or Amy Lowell on the other. In architecture the Greek temple and the Gothic cathedral. In every art the classic movement and the romantic. Likewise, in the art of philosophy (for it is truly an art) the same tendencies can be marked, and few more striking examples of this difference could be found than the pair of men of whom we are talking.

And by the by, how much richer the world is for having both types. Certainly, it is for many of us who are mixtures, who like myself may prefer Raphael to Whistler at the same time as they prefer Shelley to Dryden and worship beauty impartially in a Grecian temple or a Gothic cathedral.

But let us allow James and Poincaré to speak for themselves. James says:

"The history of philosophy is to a great extent that of a certain clash of human temperaments. Undignified as such a treatment may seem to some of my colleagues, I shall have to take account of this clash and explain a good many of the divergencies of philosophy by it. Of whatever temperament a professional philosopher is, he tries, when philosophizing, to sink the fact of his temperament. . . . Yet his temperament really gives him a stronger bias than any of his more strictly objective premises. . . . He trusts his temperament. Wanting a universe that suits it, he believes in any representation of the universe that does suit it." [12]

Could there be a clearer recognition of tastes? And again:

"I know very well that in talking of dislikes to those who never mention them, I am doing a very coarse thing, and making a sort of intellectual Orson of myself. But, for the life of me I can not help it, because I feel sure that likes and dislikes must be among the ultimate factors of their philosophy, as well as of mine. Would they but admit it! How sweetly we could hold converse together!" [13]

No quotation can give an idea of James' love of diversity. But I challenge any of you to read a dozen pages of his delight-

[12] *Pragmatism*, page 6.

[13] *Essays in Radical Empiricism*, page 275.

ful letters and not feel this pulsing joy in kaleidoscopic life. Also if you read a dozen pages you will read both volumes. I may merely add in this connection that James considered a career as an artist and even studied art for a year.

As to Poincaré, if ever a man appreciated the beauty of mathematics it was he. None expressed it better, thus:

"Mathematics has a triple aim. It must furnish an instrument for the study of nature. But that is not all: it has a philosophic aim and, I dare maintain, an æsthetic aim. It must aid the philosopher to fathom the notions of number, of space, of time. And above all, its adepts find therein delights analogous to those given by painting and music. They admire the delicate harmony of numbers and forms; they marvel when a new discovery opens to them an unexpected perspective; and has not the joy they thus feel the æsthetic character, even though the senses take no part therein? Only a privileged few are called to enjoy it fully, it is true, but is not this the case of all the noblest arts?" [14]

How dull this weak, secondhand presentation must seem to all of you! But to me the preparation of this has been a very lively experience. The contact with these two philosophies has been inspiring and it has been of great interest to trace how James was lead to philosophy through an irresistible impulse that started at college and impelled him along this road in spite of his scientific work, while Poincaré came to it as the logical completion of his mathematical works. It was the rounding off of his system of thought, the giving mathematics its proper setting. To him mathematics filled a central place in all systems of thought and thought is everything. Of that he was certain. He ends his "Value of Science" thus:

"Every act should have an aim. We must suffer, we must work, we must pay for our place at the game, but this is for the seeing's sake; or at the very least that others may one day see.

"All that is not thought is pure nothingness; since we can think only thoughts and all the words we use to speak of things can express only thoughts, to say there is something other than thought is, therefore, an affirmation which can have no meaning.

"And yet—strange contradiction for those who believe in time—geologic history shows us that life is only a short episode

[14] *The Foundations of Science* (Value of Science), page 280.

between two eternities of death, and that, even in this episode, conscious thought has lasted and will last only a moment. Thought is only a gleam in the midst of a long night.

"But it is this gleam which is everything."[15]

Bibliography

The following bibliography is in no sense exhaustive but suggests some reading either portraying the personalities or showing the philosophies of James and Poincaré.

Biographical

1. The Letters of William James. Atlantic Monthly Press, Boston, 1920.
2. Henri Poincaré. Paul Appell. Plon Nourrit and Co., Paris, 1925. (In French.)
3. Henri Poincaré. Ernest Lebon. Gautier Villars, Paris, 1909. (In French.)

   This contains a Biographical Address by Frederic Masson delivered when Poincaré joined the French Academy, also a philosophical note by Emile Faguet. (In French.)

Philosophical

1. Pragmatism. William James. Longmans Green and Co., New York, 1907.
2. The Meaning of Truth. William James. Longmans Green and Co., New York, 1909.
3. Essays in Radical Empiricism. William James. Longmans Green and Co., New York, 1912.
4. Henri Poincaré. Volterra and Others. Felix Alcan, 1914.

   Contains a chapter by Poincaré's brother-in-law Pierre Boutroux on his philosophy. (In French.)
5. The Foundations of Science. Henri Poincaré. Science Press, New York and Garrison, New York, 1913.

   This includes, in translation, Science and Hypothesis, The Value of Science, and Science and Method.

Of the above I believe the two most interesting for general reading are *The Letters of William James* and *The Foundations of Science.*

[15] *The Foundations of Science* (Value of Science), page 355.

THE FOUR FUNDAMENTAL ARITHMETICAL PROCESSES IN ADULTS

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Are the addition, subtraction, multiplication, and division combinations simple and automatic responses on the part of adults? Do they all perform these operations in a somewhat similar manner or has each developed a scheme of his own for getting the acceptable result? Such questions come to the teacher watching closely a class of adults performing the operations.

Would perhaps the length of time given by the worker to the various phases of the four fundamental processes in arithmetic tell anything about the nature of these processes? The time required by college sophomores for the various parts of addition, subtraction, multiplication, and division was measured in the hopes that some light might be thrown on how adults do the supposedly simple operations.

A kimeograph was used in making the record. One electrical current included a tuning fork vibrating fifty times a second. An open circuit was made with a brass plate on one end and a pen at the other end. Into this circuit was connected a fine marking needle. When the pen touched the brass plate, the circuit was closed and the needle indicated the writing on the smoked paper. By dropping perpendicular lines from this record to the record made by the tuning fork, the time used for each figure could be measured in fiftieths of a second. Subjects worked problems as they ordinarily would, using the brass plate in the place of the usual paper. After a while a severed connection was remade and a record taken. In this way, the relative time for the different operations was ascertained, and the reactions of the subjects observed. Careful tabulations of the records so obtained are preserved.[1] The problems used were

[1] Inga Olla Helseth, "Psychology of the Number Consciousness," Manuscript Master's thesis at Florida State College for Women.

selected from Curtis tests. Twenty-six subjects took part in the experiment, but not nearly all twenty-six were present for all parts of the work.

In general it was found in this experiment that about one half of the time spent on the problems was given to the mechanical parts of copying problem, recording answers, making figures needed in the computations, etc. Individuals varied less in the percentage of time given to the different phases of work than they did as to actual length of time given. The association time required for any type of work was far more variable in an individual than was the time used for the mechanical work of the problem.

The mental processes continued while results were being written and retarded the writing. Indeed, whenever several associated processes were before the individual at once, the work was so integrated that the time required for the processes was not that of the sum of the period required for the processes when each was performed separately, nor yet was it as little as that required for only the one process. For instance, numerous phases of work may be necessary in subtraction before putting down one figure. The operation may require merely finding the difference of two digits. It may involve the additional task of "borrowing" or both that of borrowing and of recalling whether from that particular number something has already been borrowed. If each operation was considered one association no matter what phases it involved, the average length of time for the group on each type of operation would be as follows (fiftieths of seconds):

Operation involving subtraction only	38.52
Operation involving subtraction and paying back	41.44
Operation involving subtraction and borrowing	47.32
Operation involving subtraction, paying back, and borrowing	65.79

But if subtracting, borrowing, and paying back were considered separate associations of equal difficulty, the following would be the association time in the various types of operations in subtraction:

Operations involving subtraction alone	38.52
Operations involving subtraction and paying back	20.72
Operations involving subtraction and borrowing	23.66
Operations involving subtraction, paying back, and borrowing	21.93

In this connection it is interesting that in multiplication the subjects used practically the same proportion of time in the different phases, about 60 percent for the pure multiplication, 26 percent in addition phases, and 14 percent in writing down problems to be worked. (Twelve subjects are considered in these figures.) This devotion of the same percent of time to different phases did not occur in subtraction nor in division. Some subjects required a great deal of time to "remember" if they had borrowed in the previous subtraction, or to find the trial divisor. Evidently these phases could not so readily be made automatic. Adding took very litttle more time in multiplication problems than in addition problems; this slightly greater time might be necessary because figures were not carefully arranged for adding. That this had some influence seemed probable from the way in which many subjects indicated with movements of the pen which numbers they were adding. But greater time was required for the multiplication and subtraction phases of division than for these phases when they were the whole of the problem. This was probably due to the fact that subjects could not give themselves wholly to the work of multiplying or subtracting since they were still concerned as to whether the trial quotient had been wisely chosen; this feeling extended into the subtraction work also, for often the subject was not certain until after subtraction that the quotient was correct. Indeed in subtraction, one subject carefully looked at all the digits in the subtrahend comparing them with those in the minuend before beginning work, so that she might "go right ahead." This would seem to be a seeking of an attitude for the problem whole. A number of subjects remarked that problems in subtraction involving several types of combinations of numbers, some requiring borrowing and some not, are more difficult than those problems made up wholly of either type.

The motorization in the work was frequently evident to the close observer without use of instruments with which to detect it. That which was observed in these experiments varied greatly in degree and kind. One subject showed it habitually in deeper breathing, another declared that she had no consciousness of any kind of physical response but showed it clearly in that her lower lip shaped itself as for pronunciation as the various numbers came into mind. But another whose lips showed the num-

bers said, "it seems as though the numbers said in my vocal organs before I know them myself"; another said, "I feel numbers in my tongue." There were varying degrees from those who "feel" the numbers on their lips through those who repeat inaudibly, those who whisper, and those who say "one to carry," to those who keep up a conversation with themselves. An example in this study of the last was one subject who while seeking the trial quotient murmured, "goes about eight times," "no," "well, try seven" and so on throughout the work; so intent was she on doing it both accurately and rapidly (because she conceived that to be the result which would be measured) that she paid no attention to the close observation of the investigator.

Larger muscular responses are also in college students. The list that follows gives the nine most clear cut of those observed in a group of twelve subjects working on addition with rook cards.

(*a*) Carried in addition by moving aside certain fingers as hand lay in her lap.

(*b*) Carried with motion of the head (Reports consciousness of many muscular tensions).

(*c*) Squinted the eyes. Looked to one side when a difficulty appeared.

(*d*) Tapped with the pencil. Kept time also with one foot and emphasized an error with the other foot.

(*e*) Visibly moved fingers up and down as she counted them.

(*f*) Motioned with fingers in the regular work. Waved hands slightly or snapped fingers lightly when in difficulty.

(*g*) Emphasized numbers with various head and foot movements. Put head to one side if in doubt.

(*h*) Rubbed forehead when troubled, made dots with fingers and then visualized.

(*i*) Accentuated numbers, swayed head in rhythm, showed erratic movements if troubled.

Addition appeared to be the most automatic of the processes and it is interesting to note in this connection that motorization was the most common method in this subject. Resort to audition, to visualization, and to other devices was more frequent in the other operations.

Do adults leave behind with their first childhood toys the

peculiar devices with which they first overcome their number difficulties? Some inferences with regard to this may be drawn from this study. In addition to the experiments already described, the subjects worked problems in the class, then tried by introspection to find out how they did the work. After this for a few days they interested themselves in introspection, after any chance mathematical operations that their college work required. The individuals privately recorded their findings, after which the class tabulated the written responses.

Some of the results from the investigations in addition indicate that the childish ways of working are still used in college days. The commonest difficulty in column addition was that of forgetting the number of tens in the preceding sum while making the combination of the units. Multiplication was often resorted to, at times because the subjects could not get the sums readily and at other times because grouping into multiplication meant added speed. The same is true of the method of skipping about in the column and making various combinations as of 5's, 10's, or even of breaking the given numbers so as to permit such combinations as the individual preferred. When making combinations of two digits whose sum made less than ten, five students found no particular difficulties, while 21 reported themselves as conscious of difficult combinations as follows:

Greater differences than 3 occurring between addends	1
9's and odd-even combinations	1
7's, 8's, 9's	2
7's, 8's	1
9's	6
9's, 7's	7
8's	3

In considering column addition, five students reported that they habitually made addition combinations, one that she regularly multiplied, and two that they consistently used division in order to combine the addends. When meeting difficult combinations in several column written addition, childhood's devices were still in use as seen below.

3 reported putting down dots.
14 used fingers.
4 used combinations of fingers and dots.
4 made combinations of numbers.
1 repeated addends consciously.

In general it was found that addition is not addition in every situation; without any conscious decision on the part of the subject it may become multiplication or subtraction. Almost unanimously it was found that one cannot make generalizations to cover all kinds of addition. Solving 8 plus 7 when they stand alone is not the same as 8 plus 7 when adding a column of one place numbers and neither of these is the same as making the combination when adding orally two three place numbers. The same individual may use different methods in each of these. One student, for instance, was found to have an immediate automatic verbal response to the first, made combinations in tens for the second, and visualized for the third type of problem mentioned.

These differences were more noticeable when the process was changed. For instance, one subject counted on dots when a difficulty in addition appeared, she visualized in subtraction, and repeated her tables from the smaller combinations up when the difficulty arose in multiplication.

When these subjects were asked to do problems with rook cards, much the same phenomena appeared. The subject handled the rook cards as rapidly as possible, adding as each number appeared, or subtracting. Individuals in this also differed as to their ways of getting results; each individual again also had different ways of making combinations, depending on the setting in which the combination appeared.

Evidently children's difficulties in the simplest mathematical operations persist into adulthood. Most cumbrous habits have in many instances been built up and still continue in use, with or without the consciousness of the individual. One subject remarked casually, on looking at a problem having a certain digit repeated many times, "I nerve myself to meet that number the moment I glimpse it in an addition column." To one, numbers are still decorated with dots, each with its appropriate number. On these she had counted as a child and still counts in a difficulty or under mental stress. She had not been allowed to count on her fingers, she said, but seemed surprised that no one else ever used dots; it had never occurred to her that other people did not count in imaginary dots on numbers. This subject said she had not succeeded in getting a satisfactory scheme for four, she could not place the dots on the figure four, re-

marked that any combination with a four in it was difficult anyway. Another subject was conscious whenever she encountered a difficult multiplication combination of changing habitually from that form in which it appeared later in the tables to that form which it first had in the tables. Nine times seven, for instance, became seven times nine before the work could proceed smoothly. Counting by feeling innervation in the finger tips is common. One subject limited hers to three fingers, why she does not know, but she visualized these three fingers moving. Another confined herself to three fingers but actually moved them as she counted. Little peculiarities came constantly to the surface. One said that she thought of numbers always as dollars; it made them more interesting. To another, each number had its own color though these were becoming less clear now than in childhood. A student who worked by visualization in addition says she has never seen a naught; further, if a number ended in a figure one, she had difficulty in manipulating it. One subject said that she tried out addition combinations when subtracting to see if any fitted, because she found subtraction difficult. In addition difficulties another "feels around" by motorizing chance sums until the familiarity of some combination satisfies her. Another spoke of herself as a "slave to counting" because at the slightest doubt, fatigue, or interruption, she falls back to counting unit by unit.

Various movements indicated any nervousness which subjects felt while at work. These movements might be slight tapping, the swinging of a crossed leg, the trembling of a hand, or even the twitching of the whole body. Several recognizing such conditions in themselves stretched muscles and settled to steadier work, all without pause in the work they were doing. Probably half of the subjects accentuated with head or pen or some other type of movement when they came to a difficulty. Very frequently the shaking of a foot or hand indicated a check in the smooth flow of the calculations, giving way to more pronounced movements over greater area if thought became entirely blocked; while a shake of the head was almost certain to appear as the subject recognized a certain or possible incorrectness. Often this shake of the head was repeated for a number of times to the end of the problem, although only one error had been made or had been felt by the subject.

The subjects did not seem to distinguish consciousness of fact of correctness from the emotional tone which accompanies the fact. In addition, for instance, some ways in which subjects know their answer to be correct are as follows: "Rush of blood comes, if wrong," "Error is like a discord when playing a piano," "Satisfaction comes when placed in number frame," "When number is whispered," "When rhythm continues," "Incorrect sum would bob up again," and errors bring "catch of breath and feeling as though falling."

As a check on results obtained in introspection during mathematical operations the same students were closely observed as to motor reactions while making effort at mastery of an association test in words. Students were asked for written introspections immediately after work was finished. More than half showed movements of lips and throat at close distance from observer. One student closed eyes and pressed hard upon them. She wrote that she pictured page and then read from memory of it as she completed test. Another closed eyes, seemingly tighter and tighter; claimed that she saw pictures of word appear. Another snapped her fingers and shook her hand when the word she desired failed to appear. Another check study was made with children in the fifth and sixth grades in connection with their difficulties located by the Curtis practice tests. Phenomena like those described in this study of adults were also found in the investigation of the children's work.

Three cases of individuals with number frames [2] were found during the investigation. One subject had learned to count, to comprehend numbers and their relative values on hers. When numbers are heard now, she sees the frame and the numbers definitely located thereon. The frame is not however used during written operations with numbers. The frame appears in a dark gray on a lighter gray background some fifteen feet out before the subject. In form it might be described as a series of wave pictures. It is peculiar in that the first break in form is at 12, thereafter at the tens. This is likewise true in the hundreds, the first high point being at 112, thereafter at 120, 130, etc., for the one hundreds for instance. The student when questioned recalled that she had learned well the number combinations up to 12 before she operated with the numbers beyond.

[2] Diagrams of these number frames are on file in manuscript previously mentioned.

The second number frame had much the appearance of "writing in white ink upon the black paper of a photograph album." In this frame 10 and 20 are important points, above 20 the figures are much more closely packed. Furthermore the hundreds are emphasized by being seen as larger and being starred. Although the frame had never seemed new, the subject's first memory of using it was in counting for the game of "Hide and Go Seek."

The third frame is about 28 inches long and two and a half high on "foggy" gray background about 15 inches before the face. Only the parts needed appear, almost always about twenty numbers. The subject described the sensation which comes upon hearing a number, as that of "finding myself placed on the frame." There are peculiar breaks at 6 and 11 which she accounts for from the fact that she learned her numbers up to 5 long before she learned any beyond and then to 10 before she had any conception of those beyond.

The frame is used in many of the operations. Numbers placed on it before operations are moved together to critical points during the operations; result is also finally placed on the frame. The mother and little sister of this subject also have number frames, but the brother when questioned laughed the very existence of such things to scorn. This subject spells on a form, and has peculiar frames for many ideas such as days of the week and months of the year; to her it is "thinking."

The whole study suggests that the adult's response to a situation has often been experimentally built up by the individual through the integration of many different reactions to many concrete lines of stimulation. These responses of adults differ widely. Not only so, but many features of these differences consist in exceedingly cumbrous devices, probably built up by trial and error method to get around mental difficulties met as a child. Later these devices were made automatic without the individual realizing their waste or their absurdity.

A MATHEMATICAL CONTEST

By HELEN M. WALKER,

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The mathematical contest described in this article was originally devised for the entertainment of the Twin City Mathematics Club by the joint efforts of Mr. Myron F. Leslie and the writer, both of whom were then teaching in North High School, Minneapolis, Minn. The Twin City Mathematics Club, a joyous and enthusiastic organization under the leadership of Professor W. D. Reeve, had the custom of devoting its June meeting to an evening of pure fun. At one of these meetings, this contest provided so delightful a bit of entertainment that it is offered here in the hope that it may prove equally interesting to other informal gatherings of teachers and to high school and college mathematics clubs.

The original list has been somewhat augmented by additions made by both Professor Reeve and by the writer, but the possibilities are by no means exhausted. It is to be hoped that teachers who make clever additions or substitutions will share them with others through the pages of this magazine.

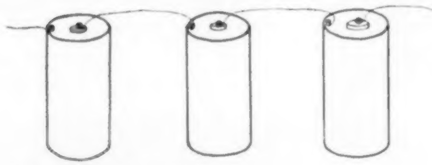
When each guest had received a pencil and a sheet of paper bearing the numbers from 1 to 42, the following instructions were read: "Distributed about the room is a mathematical exhibit, each article of which represents a familiar term taken from the field of elementary mathematics. Each article in the exhibit is numbered. Write the term represented by an article opposite the corresponding number on the paper provided."

The articles used in the exhibit were as follows:

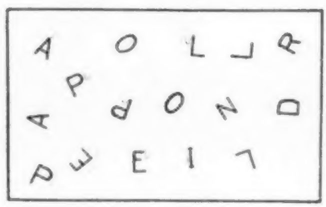
1. A basket of potatoes, onions, carrots and radishes.

2. | | |
|---|---|
| NO | NEVER |
| NOT | NONE |

3. A red flag.

4. 

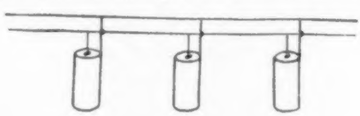
5. A cube cut from a carrot.

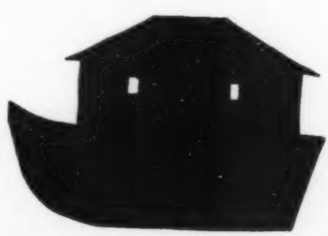
6. 

7. 8 by 4 by 4.

8. A pair of cymbals, or a picture of the same cut from a catalogue of musical instruments.

9. Several cubical blocks lying on an otherwise bare table.

10. 

11. 

12. A pair of brackets, or rough sketch of them.

13. Several large sticks of stovewood.

14. A signboard, or drawing of one.

15. A page torn from the index of a book.

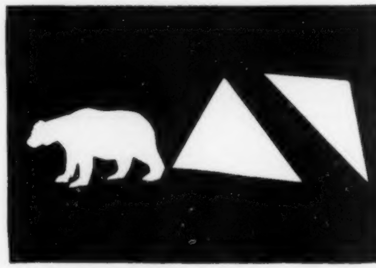
16. A collection of such musical instruments as the bassoon and bass viol, or pictures of them.

17. NO MORE PRIVATE PROPERTY!
ABOLISH ALL CAPITALISTIC GOVERNMENT BY FORCE!

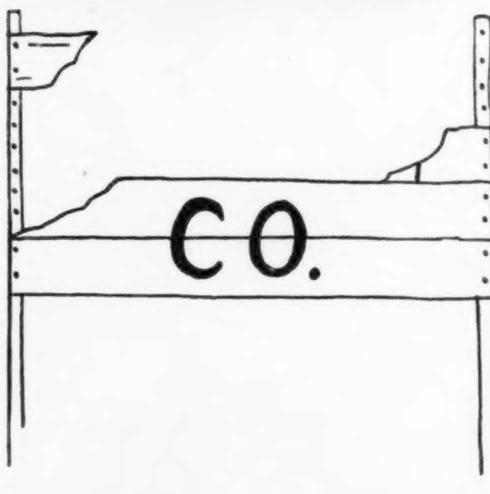
18. Pine cones.

19. $\boxed{H_2SO_4}$

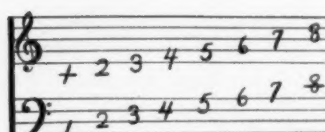
20.



21.

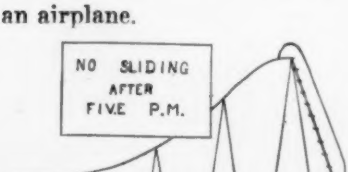


22.



23. A picture of an airplane.

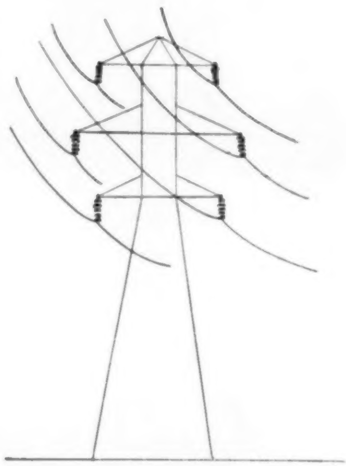
24.



25.

M.A.	D.D.S.
Ph.D.	D.D.
M.D.	S.T.D.
J.D.	LL.D.

26.



27. A signboard with the inscription "First Prize" and a hand pointing to a large onion.

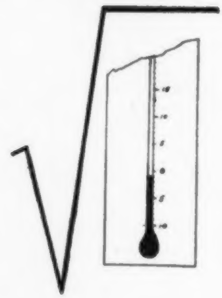
28. A typewritten sheet reading as follows:

    Extract from the Montana Code. Statute No. 973. Be it hereby enacted: That all cities of the first class shall be granted and accorded power and authority to declare illegal the display of SIGNS and BILLBOARDS for advertising purposes along the streets of said city of the first class, and shall be furthermore empowered and authorized to levy fines for such misdemeanor against the owner of the property upon which the aforesaid SIGNS and BILLBOARDS may stand, said fine not to be in excess of $50 for each offense.

29. A clipping from the advertising section of any newspaper.

30. Pictures of any European monarchs, past or present.

31. A collection of photographs so cut as to show the faces only.

32. 

33. A picture of a monkey.

34. A picture hanging upside down.

35. Several pictures of a ship's compass, such as may be found in a Keuffel & Esser catalogue.

36. A placard marked "Look" with an arrow pointing to a picture of Immanuel Kant.

37. A rustic table built from logs.

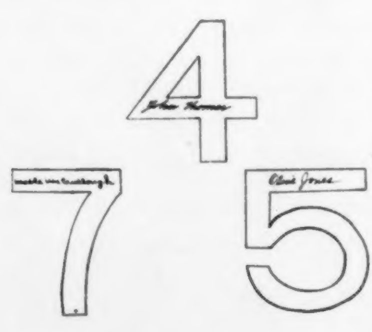
38. Pictures of Trotsky, Lenine, Emma Goldman, Eugene Debs, Saklatvala, etc.

39.

Village of PLEASANT VALLEY Slow down to 20 miles an hour.

40. Two or more axes.

41. A package of oleomargarine.

42. 

KEY

1. Extracted roots.
2. Negatives.
3. Radical sign.
4. Series.
5. Cube root.
6. Pi.
7. Chord.
8. Symbols.
9. Table of cubes.
10. Parallel.
11. Arc.
12. Brackets.
13. Logs.
14. Sine.
15. Index.
16. Bases.
17. Radical expressions.
18. Cones.
19. Solution.
20. Polar triangles.
21. Cosine.
22. Number scale.
23. Plane.
24. Slide rule.
25. Higher degrees.
26. High power.
27. Indicated root.
28. Law of sines.
29. Add.
30. Rulers.
31. Faces. ("Simple expressions" has been suggested.)
32. Imaginary number.
33. Evolution. (Also credit origin.)
34. Inversion.
35. Compasses.
36. Secant.
37. Table of logs.
38. Radicals.
39. Limit.
40. Axes.
41. Substitute.
42. Signed numbers.

A CRITICAL EVALUATION OF INDIVIDUALIZED INSTRUCTION IN MATHEMATICS

By WILMA SHAFFER FLEWELLING

Emerson High School, Gary, Indiana

Now that there is no longer a question in the minds of teachers that teaching should be adapted to the individual and that the needs of individuals differ, each good teacher aims, as far as possible, to reach his pupils individually.

Attempts have been made to devise modes of instruction which will allow the needs of each pupil to receive fuller individual recognition.

A suggestive and detailed account of individualized instruction in seventh and eighth grade arithmetic in the Winnetka Schools is given in THE MATHEMATICS TEACHER for December, 1922. In Winnetka, goals for each unit are formulated. The material is self-instructive as nearly as possible. In the lower grades, there is some oral developmental work while in the upper grades the developmental work is in the practice books with the exercises. Under this system a child cannot fail. He never repeats. Each pupil accomplishes as much as he is capable of accomplishing. At the end of the month, a goal book (showing the number of goals reached by each pupil) is sent home for the parents' inspection and signature.

An interesting account of individualized instruction in geometry is found in THE MATHEMATICS TEACHER for April, 1926. In this procedure, no textbooks are used; the pupils make their textbooks during the course. Each unit is introduced by developmental lessons. By this means, the way is paved for individual effort later. A child then progresses through each unit according to his own ability. A review and test follow each unit. Students after mastering a particular assignment take care of checking and recording progress on an individual record sheet. As to results, the teachers are agreed that there are better returns. Three fourths of a group of students, more than half of whom had an I.Q. below 110, exceeded the standard median of the Schorling-Sanford Geometry Test.

Individualized instruction in ninth year algebra is discussed in THE MATHEMATICS TEACHER for April, 1925. In this experiment each pupil is promoted in each part of the work whenever he completes that unit. The work is divided into small units called goals which have to be attained by the slowest pupil. There are within each unit practice materials corresponding to each weakness likely to be encountered. All practice exercises are self-corrective. Tests are complete and diagnostic. No homework is required. As to results, a class of eighteen pupils whose average I.Q. was 93 completed the second semester of algebra without a single failure. Some pupils completed the minimum course as much as eight weeks before the end of the term, thus making it possible for the course to be enriched with supplementary materials.

Reports of actual class room attempts along this line are few. The essential characteristics of the technique of the individual method may be summarized as follows:

1. The assignments (covering an entire unit of work) is made in advance.
2. The pupil works up to his capacity.
3. The class exercise is modified as the work progresses. At the outset, all of the pupils are together; soon the class is broken up into groups; these in turn are subdivided until sooner or later very few if any pupils are together. Explanation and assistance are given to groups when possible; to individuals whenever needed. Some teachers have pupils assist each other to some extent during the class hour as well as outside; others conduct the class work much as usual; still other teachers abandon the class exercises entirely and spend the class hour in passing about the room answering the questions of the pupils, each of whom is working on his own problem.
4. The work assigned is divided into units. At the close of each unit, each pupil is tested. According to the outcome of each test, he is assigned supplementary work to do on that unit; or if the results of the test are satisfactory he is promoted to the next unit.
5. When the pupil has thus completed the work which is required, he is excused from further attendance in the class and credited with the subject. The quickest pupils usually finish in about half the time allotted under the class system.

The advantages and disadvantages that have been listed by teachers may be enumerated as follows:

Advantages

1. There is a fair chance for each pupil, especially the slow pupil who would be crushed beneath the competition of the class.
2. Each pupil works all the time at his own rate.
3. Self-reliance is cultivated in the pupil.
4. The pupils do more thorough work.
5. They do more work.
6. Whatever is well done need not be repeated, even though the pupil cannot complete the work. He begins next year or term where he stopped last. Waste is prevented. The work is thorough as far as carried.
7. The weaker pupils are not carried beyond their depth.
8. The pupils are more sincerely interested in the work.
9. There is a more cordial feeling toward the teacher. She is a friend in need, not a taskmistress—or even a drill mistress.

Disadvantages

1. The benefits, to the members of the class which are derived from social participation are lost.
2. There is a tendency to superficial work by those in haste.
3. Some pupils lag. They need the stimulus of class exercises.
4. The incessant changes from one part of the subject to another and the unrelaxing alertness required to seize and handle well and quickly the diverse special needs of the pupils are wearing in the extreme upon the teacher.
5. With classes of twenty-five to fifty pupils, the teacher can give to each pupil a maximum of two minutes time per class exercise.
6. If a teacher wishes to keep in adequate touch with the progress of the pupil, she must require an amount of written work entirely too large for her to correct. The preparation of test papers for each pupil (no two papers alike) is in itself a serious task.
7. The preparation of an adequate amount of drill and remedial work is hardly possible in the case of a teacher carrying her allotted load.
8. There is a tendency also to neglect the application of the

social side of the subject, and to give almost exclusive attention to the mechanical side.

It seems to me that the disadvantages enumerated above may be well taken. Is it not possible to work out a sane combination whereby all the advantages may be secured as results without adhering to individualized instruction in its extreme form?

Morrison in his studies in high-school procedure shows that the lesson learning procedure (and there is this type of learning if the individual is working alone) does not function. That is, the learner is unable to apply the knowledge. He also shows that the idea of covering ground is another serious handicap. Mr. Morrison offers as a solution this thesis: What is worth doing at all is worth doing well. He says that mastery in school work does not involve any serious difficulties, but that it does involve a new conception of teaching. It involves ceasing to measure a pupil by his average mark and measuring him by the quality of his achievement.

If a procedure has mastery as its objective (and to me it is the sane objective), what provisions can be made for differences in the learning rate?

In this case, experiments carried on so far have shown that such differences are not nearly so marked as in the case of a lesson learning procedure. If the theory is sound, mastery by all pupils (save the genuine problem cases) should eliminate a large class of marked differences in progress through a given unit. As training goes on, the rate of learning of the majority of the class seems to be materially nearer together than formerly. None the less a few individuals are always conspicuously more rapid learners than the majority of the class, and a few others are capable of much more and much better work. To meet the situation a plan of excess credit work was suggested.

The question next arises: How can this thesis and this procedure apply in the teaching of mathematics and still allow the needs of each pupil to receive fuller individual recognition?

The following procedure is suggested for plane geometry first, teaching objectives should be clearly in mind—in every case the objectives will be a principle or a body of principles to be understood or a power to be gained; second, the course should be outlined in manageable units. These units will be units of understanding rather than pages or chapters to be covered. Third,

this formula should be used in the procedure—teach, test, and teach again if necessary.

In outlining the subject matter for each unit, the minimal amount of material as outlined in the report of the National Mathematics Committee can be used. Enough concrete applications to develop the principles or powers outlined for the particular unit should be provided. Then teach, test (remembering that a test is always a means of measuring teaching), and teach again if necessary.

The excess credit work for each unit should also be outlined. This will consist of the additional material as outlined by the mathematics committee with concrete applications of these additional facts. Also more difficult concrete applications of the minimal material should be included. Additional material in each assignment should be provided.

The question of checking may be taken care of by having a cardboard on which is each pupil's name. The cardboard should be divided into little squares, each square representing a unit of the excess credit work. As the pupil finishes a particular unit, the square is made red. The pupil can compare his progress with other fellow students.

Of course this will call for a great amount of work, but not so much as when each is working along at his own speed and the teacher is attempting to keep a close check on each individual.

The test following each unit should be divided into two divisions. The first division should be based on the material necessary for the mastery of the particular unit; the second division should be based on the extra credit work.

The students should be inspired to work on the material required for a mastery of the particular unit before going to the extra credit work. This will keep the majority of the class busy. The very slow pupil can easily be found during the directed study period. He should be taken under the teacher's wing and given extra help and encouragement.

To me the disadvantages of the method of individualized instruction in its extreme form outweigh the advantages. Following the procedure outlined above, the pupils will benefit from the class hour, the stimulus from the class exercises will serve as a spur for the student who is prone to lag, the personality of the students will be given more opportunity for expression, more

time will be given the pupil by the teacher through developmental work, directed study, and summaries, and lastly *the teacher will be a teacher and not a clerk.*

There is a tendency to superficial work and so-called "leeching" on the part of some students. This occurs in all methods of teaching and can be overcome in one as well as the other.

What is the greatest claim of each pupil as an individual upon his teacher? Is it that he masters a principle or a body of principles to be understood or a power to be gained, or is it that he may come into as close and constant touch as possible with his teacher's personality, scientific and individual, obtaining from his teacher the maximum, not only of knowledge, but of guidance, of stimulus, of inspiration as well? These two are not mutually exclusive, quite the contrary.

It seems to me that the plan outlined above will do more to realize these two claims than individualized instruction in its extreme form, while at the same time it will sufficiently care for the needs of the individual.

THE LABORATORY METHOD IN TEACHING OF GEOMETRY

By C. A. AUSTIN

Venice High School, Los Angeles, Cal.

About ten years ago a group of experienced teachers of mathematics in Fresno, California, realized that it was absurd for young people to be "addling their brains over mere logical subtleties, trying to understand the proof of one obvious fact in terms of another, equally, or it may be, not quite so obvious, and conceiving a profound dislike for mathematics, when they might be learning geometry, a most important and fundamental subject which can be made very interesting and instructive."

These teachers caught the idea that plane geometry is "essentially an experimental science, like any other, and that it should be taught observationally, descriptively, and experimentally in the first place." They proposed that plane geometry is a laboratory science, and that, as such, it could be taught to children more effectively and satisfactorily than by the old Euclidian plan. They argued that the inherent nature of the subject-matter demands a scientific and experimental treatment; that the child to whom the subject is to be taught is fundamentally a scientist, who lives and learns by experimentation and observation in a wonderful world laboratory; and that the social and economic needs of children require a practical and sensible presentation of the material.

This small group of teachers, after a year or two of experimentation, boldly undertook, in their humble way, the problem of teaching plane geometry as a laboratory science. They attempted the solution of this problem, first, by establishing worthy and obtainable objectives; second, by selecting and organizing the subject-matter; third, by devising methods of attack and procedure and teaching technique suitable to the ability and needs of the children; fourth, by adjusting materials and methods to the individual capacities of the pupils; and fifth, by developing good habits of study.

Plane geometry, they said, must be so organized and presented that it will conform to modern ideas in educational thought and that it may meet the social and economic demands of life. In order to accomplish this purpose, definite and tangible objectives were formulated.

The subject-matter of plane geometry has remained almost constant for centuries. Many useless and complicated propositions and problems, however, have been eliminated in the process of adapting the materials to the capacities and needs of children. The real task before the teacher is not so much the addition or subtraction of material as it is the organization of the subject-matter into a unified whole. There is an unbroken chain of closely related unit-links extending throughout the subject. Each link has a definite position and function in the chain. Each link is a unit within itself, for it consists in a group of several elements organized about some unifying principle. The teacher or text must group the details of the material about these organizing principles, and the class should be aware of the fact that each lesson deals with some phase of a general problem.

Attack, Procedure, and Teaching Technique.—Upon this phase of the problem these teachers devoted much study, research, and experimentation. After establishing the idea that plane geometry is a laboratory science, the avenues of attack, procedure, and teaching technique were opened up and paved. Their mode of procedure brought out in a new and clearer light such items as constructions, definitions, theorems and corollaries, demonstrations, recitations and assignments, tests and measurements, reviews and summaries, laboratory apparatus, and outside activity.

The Laboratory Idea.—The keynote of the laboratory idea is discovery by means of experimentation. Pupils should be permitted to observe the laws of geometry operating in concrete form before they are required to do logical thinking. This can be done if they are set at work with paper, pencil, compass, protractor, straight edge and scale, and directed scientifically in their construction of figures and in their study of those figures. They may thus be led into the spirit of investigation and discovery and into habits of productive methods of procedure.

The fundamental process is directed activity with objective

materials. Interest is awakened, for the pupils see real points, lines, angles, and figures with their eyes while they are making them with their own hands. There is motivation for they are engaged in purposeful activity. Reflective thinking, which accompanies their directed study of the concrete embodiment of geometric principles, leads them to the discovery of facts and proofs. There is, therefore, satisfaction in accomplishment.

To accomplish this is a very simple task for the teacher. All he need do is to set pupils at work with proper tools on carefully prepared directions and then guide their activities. If a text, which contains directions for constructions and study, is used, just place it in their hands and let them alone. Teach them to read and to do step by step and in order what they are told. The teacher, however, will find himself quite busy answering questions, offering help and suggestions, and assisting pupils in judging the value and correctness of their product.

The steps in pupil activity, according to the laboratory plan of teaching plane geometry, are as follows:

1. He makes constructions according to specific directions, which are so prepared that he is led from a definite point of departure through logical steps in proper sequence, to a desired result.

2. He takes measurements on his constructions and performs computations, whenever possible, to lead him to desired conclusions.

3. He makes an analytic study of the conditions of construction in connection with the desired results to discover an apparent truth or definition.

4. He states the definition, or gives the apparent truth as a theorem, and then prepares the demonstration to establish it as a general proposition.

5. He applies this truth to many exercises and problems.

Constructions.—A key to success in teaching plane geometry is construction. This instrument unlocks the doors of activity, interest, motive, and understanding. The first task, therefore, is to get pupils busy doing purposeful things with their hands. When this is accomplished—and it is not difficult to do so—many of the initial troubles will be avoided. Constructions that are simple, well organized, and purposeful develop skill in performance and satisfaction in accomplishment. A large fund of

information will be deposited in the mental bank to be drawn upon when needed. A scientific method of approach will thus be produced. The natural instinct within children to see how things are made and put together and also the nature of the subject-matter itself demand this mode of treatment.

There is as much science and method in the construction of a geometric figure to satisfy given conditions, no matter how simple it may be, as there is in the construction of a building or bridge. There is just one thing to do first, just one thing to do second, etc., until the finishing touch is made. There is system and logic in it all. A text should be so prepared that pupils may follow explicit directions in their construction and study of figures that they may discover this scientific procedure and become skillful and accurate in performance.

The correct reading and interpretation of English is essential. Much practice in construction work will develop the power to undertake an independent exercise or problem in a scientific and therefore a fruitful manner. Time must be taken to analyze the statement of a problem so that the steps in the construction and the order in which they should be taken may appear.

Definitions.—Definitions are primary and fundamental for scientific investigation in plane geometry. There is usually just one true definition for any one thing, and the statement of that definition can be correctly made and learned. Why accept any statement, which is not a real definition, just because it happens to be a pupil's own words? Is it a hardship for the pupil or is it a transgression of any psychological law to require pupils to learn the exact words of a definition?

If a definition is presented scientifically, a correct statement of it will come perfectly natural to pupils. If, on the other hand, the pupil is required to learn a definition from a book, without any introductory experimental study, the definition has little or no meaning; and frequently it is difficult to convince him that some descriptive characteristic is not a definitive attribute. Observations and a scientific study of real things to discover the attribute or attributes which differentiate one class from another should be made before the name of the class or definition is given.

Correct definitions are the foundation upon which demonstrations are built. If pupils do not comprehend and learn a correct definition, they may never know why there is all the *ap-*

parent confusion in a proof. In fact, they seldom realize the necessity for a demonstration. They are too frequently told—not scientifically led to see—that a theorem is a statement that requires proof. This they accept on faith, for they are not able to dispute it. Proofs are, therefore, unnecessary and meaningless to them.

For example: If congruent triangles are defined as triangles which can be made to coincide, then when one triangle is constructed so that two sides and the included angle are equal to two sides and the included angle of another given triangle, it is evident that the two triangles are not made congruent; and if they are congruent it is necessary that they be proved congruent by making them coincide. Thus the pupil sees the origin of a theorem and the necessity for a proof.

Theorems and Corollaries.—The first real problem in teaching plane geometry arises the moment theorems and demonstrations are introduced. These difficulties can be eliminated, however, by a laboratory method of approach.

The initial step in a scientific approach to a theorem is to develop by constructions and discussions the conditions giving rise to the theorem. It is absurd to present, first of all, the word theorem and the fact that it is a statement requiring proof, the statement of the theorem in the book, and finally the formal demonstration. These things should be the natural results of a laboratory development, so conducted that pupils may realize what a theorem is even before the word is given. It is best to study the real thing objectively, experimentally, and observationally even before it is named, defined, or stated. According to the laboratory plan, therefore, the real origin of a theorem appears, and the fact that it must be proved is evident.

To illustrate, take the subject of parallel lines. First, a clear-cut definition of parallel lines must be developed. Do this as follows: Draw on the board or on paper two lines, *AB* and *CD*, that seem not to intersect however far produced. What is the condition considered in drawing these two lines? Do they lie in a plane? What is the name given to such lines as *AB* and *CD*? Must they lie in a plane? What are the two tests of parallel lines? First, they must lie in a plane; and second, they must not intersect however far produced. What are parallel lines? Parallel lines are lines in a plane which do not intersect however far produced.

To experiment further with the definition of parallel lines: Draw line *AB* on the board. Select point *P* not *O* on *AB*. Propose this problem: How can we make a line through point *P* parallel to *AB*? Take a straight edge, place it against the board, and revolve it on point *P* until it appears not to intersect *AB* in either direction. Now draw line *CD* through point *P* along the straight edge. This line is made parallel to *AB*, because it is in the plane of the board and because, in our judgment, it does not intersect *AB* however far produced.

Next, introduce the drawing necessary to develop the meaning of transversal and the pairs of angles formed.

Now, proceed as follows: Draw line *AB* on the board. Select point *P* not on *AB*. Propose this problem: How can we *construct* through point *P* a line parallel to *AB*? Draw a line through point *P* and intersecting *AB* at *C*. Draw line *MN* through point *P* so that an interior angle at *P* is *made* equal to the alternate-interior angle at *C*. When the construction is completed, let the development be as follows:

Question. Was *MN* made parallel to *AB*?

Answer. No, sir, it was not.

Q. How was *MN* constructed?

A. *MN* was constructed so that a pair of alternate-interior angles are made equal.

Q. But is *MN* parallel to *AB*?

A. Yes, sir, it appears to be.

Q. If *MN* is parallel to *AB*, what must be done?

A. We must prove it.

Q. How can we prove it?

A. By showing that the two lines lie in a plane and that they do not intersect however far produced.

Q. Now, what statement of truth about the results of this construction can be made?

A. Two lines in a plane cut by a transversal are parallel if a pair of alternate-interior angles are made equal.

Q. What is the name of such a statement?

A. It is a theorem.

Q. Why is it called a theorem?

A. Because it is a statement that requires proof.

The Recitation.—The recitation is real school activity. The

success or failure in plane geometry depends largely on the manner in which the recitation is made to function.

The word recitation implies a "recital" of material before a teacher. A mere "rehearsal" of a demonstration committed to memory, or a reading of a proof copied on the board from a book, is usually worse than a waste of time. Such a recitation fails to measure up to the requirements set by the objectives, since it places pupils in a passive, uninterested, and inattentive attitude, and since it makes the teacher a mere judge and critic. This form of lesson is not worthy of additional comment.

The recitation, however, in which there is purposeful and directed activity, in which there is freedom in the interchange of thought, and in which there is a logical procedure aiming at the solution of a definite problem, is the ideal form of a lesson required in the laboratory method of teaching plane geometry. In this way the teacher is truly a leader and director of a course of thinking. Each pupil not only follows eagerly, but he projects his thought in order to make contributions.

Let the pupil stand before the class—and he will do so without being required if there is proper stimulus and motivation—and present and defend his point. Let him stand before the class—not the teacher—and discuss—not recite. If the scientific or laboratory method is followed, each pupil will know what he is talking about for he has done it with his own hands. There will be an atmosphere of scientific authority in his discussions.

Each recitation should be the solution of a problem. Let each pupil give his own solution in full, whether right or wrong; and then the listeners should be given an opportunity to ask questions, to make changes, to correct errors, and to pass judgment on the correctness of the solution. If this is done, the entire class will be alert, active, and critical.

The recitation must be directed by a skillful teacher along logical channels and carefully guided from a definite point of departure to a specific conclusion, avoiding unnecessary debate, contradictions, and aimless wandering from the line of thought. Too frequently a series of thoughtless contradictions is mistaken for a good argument. The teacher should always control the drift of the discussion and bring the wandering minds back to the point at issue whenever it is necessary.

The lecture type of recitation should be employed as early as

possible. This may be done with exercises and originals. Let a pupil step out before the class, state his problem, make his drawing and discuss it as he constructs it, and then prove his point. To develop the ability to do this is no easy task, it is true, but it is an ideal to which one may strive.

The assignment is a vital factor in the success of a recitation. It should be stated, if possible, in the form of a problem. This may be difficult to do at times, but the ingenuity of a wise teacher will show the way. The assignment should never be made in a hurry. Sufficient time should be taken to state it clearly and to start pupils working on it so that there may be a motive and an interest in doing it.

Adjusting Materials and Methods.—The subject-matter of plane geometry and the method of presentation must be adjusted to meet the need and capacities of children. Individual differences are quite evident in any class in geometry, even in the first few recitations; and if these differences diverge and become more pronounced at the close of a semester or year, there is nothing wrong. This is perfectly natural and must be expected.

In any ungraded class in geometry pupils will, in general, take positions in one of three levels: namely, (*a*) the weak and below average, (*b*) the average and above average, and (*c*) the superior. In large schools where the number of sections of geometry permits, classes may be graded into what is commonly known as "*X*," "*Y*," and "*Z*" groups. The content of offerings for each group must differ. For the "*Z*" group the content should be broad and not very deep; for the "*Y*" group it should be narrower and deeper; and for the "*X*" group it should be still narrower but deepest.

Group Offerings in Geometry.—The teacher must select the material in plane geometry, if the text does not do this satisfactorily, to meet the needs and abilities of pupils in each group. If the laboratory plan is followed, it is possible to have each pupil, in whatever group he may be, working to his full capacity. The adjustment of the content of group offerings may be represented graphically. See the *California Quarterly*, October, 1926, page 30. (A reprint of this article, "Laboratory Plane Geometry," may be secured by writing to Scott, Foresman Company, Chicago.)

Habits of Study.—The first and perhaps the most important task for the teacher is to teach pupils how to study geometry. The subject-matter is new and strange, the language of the text is somewhat technical and quite different from that with which they are familiar, and the drawings are sometimes puzzling.

Supervised Study.—Supervised study, wisely used from the very first lesson in geometry until it is no longer needed, is a solution of the pupil study problem. The teacher should read the text with pupils in class at the beginning, assisting them to do the things therein required, and help them to interpret, understand, and learn. This should be done the very first period and for as many periods thereafter as the teacher deems necessary, whether the schedule provides for a period of supervised study or not. The assignment for home study after such a period of supervised study may be made up of directions for the preparation of the material studied, neatly, accurately, and in the required form.

During supervised study periods the teacher should study with the pupils, so that they may acquire independent and productive habits of investigation. They must be taught, first of all, to read English accurately and to make the correct interpretation of what they read. They must utilize the textbook by following the directions explicitly; by reading all hints, suggestions, demonstrations, and solutions carefully; by looking up all references; and by learning the meaning of new words, expressions, and symbols. Pupils must provide themselves with the necessary geometry tools and have all materials at hand with which to work while they read the text. They must visualize every item described even if it is necessary to make a construction to assist them. It will never do to pass over a single item that is not understood.

In conclusion let us say that the laboratory method of teaching plane geometry encourages pupils to think for themselves, to judge results, to concentrate, and to ask for assistance only when they have exhausted their own power. This plan of teaching geometry assists pupils to acquire the spirit of independent investigation through observation and experimentation, and the teacher should rejoice with them in their joy of discovery and accomplishment.

ATTENDANCE AT DALLAS MEETING OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

English, Harry	Washington, D. C.
Austin, C. M.	Oak Park, Ill.
Johnson, Mrs. Elsie P.	Oak Park, Ill.
Schreiber, Edwin W.	Maywood, Ill.
Slaught, H. E.	Chicago, Ill.
Taylor, E. H.	Eastern, Ill.
Knight, F. B.	University of Iowa, Iowa
Kelly, Mary	Wichita, Kansas
Magnuson, Amanda	Lindsborg, Kansas
Phillips, A. W.	Emporia, Kansas
Weimar, M. Bird	Wichita, Kansas
Simmons, Eugenia	Shreveport, La.
Evans, Geo. W.	Lynn, Mass.
Utley, Ruth H.	Michigan
Haertter, Leonard	St. Louis, Mo.
Huff, Louise H.	St. Louis, Mo.
Clark, John R.	New York City
Reeve, William David	New York City
Bowers, Jacob	Columbus, Ohio
Emswiler, H. W.	Columbus, Ohio
Gugle, Marie	Columbus, Ohio
Hammond, Jane M.	Columbus, Ohio
Hammond, Rose L.	Columbus, Ohio
Marquand, C. B.	Columbus, Ohio
Marquand, Helen H.	Columbus, Ohio
McMillan, H. L.	Columbus, Ohio
McMillan, Mapel Dell	Columbus, Ohio
Pollock, C. A.	Columbus, Ohio
Preston, Amy F.	Columbus, Ohio
Smith, V. M.	Columbus, Ohio
Wager, W. W.	Columbus, Ohio
Bogue, Mattie E.	Okmulgee, Okla.
Bulcher, H. P.	Ada, Okla.

Corder, Alta L. .... Cordell, Okla.
Hassler, J. O. .... Norman, Okla.
Hawkins, Julia Lee .... Chickasha, Okla.
Walker, Burney .... Okmulgee, Okla.
Wax, Edgar .... Edmond, Okla.
Zant, James H. .... Durant, Okla.
Adams, B. T. .... Wichita Falls, Texas
Andrews, Nannie D. .... Dallas, Texas
Angel, Minta F. .... Dallas, Texas
Austin, Mrs. Jewel .... McGregor, Texas
Baker, Beulah .... Dallas, Texas
Barclay, Mrs. Annie .... Hillsboro, Texas
Bedford, Virginia .... Thornton, Texas
Bennett, Clyde .... Lewisville, Texas
Black, R. C. .... Campbell, Texas
Bledsoe, J. M. .... Commerce, Texas
Boley, T. R. .... Ft. Worth, Texas
Bramlette, Mrs. Minnie .... Dallas, Texas
Brewer, Margaret .... Mineral Wells, Texas
Brooks, J. R. .... Nacogdoches, Texas
Brown, Emma H. .... Dallas, Texas
Brown, Myrtle C. .... Denton, Texas
Brundage, Daisy .... Ft. Worth, Texas
Cain, E. M. .... Dallas, Texas
Carter, Jacob W. .... Dallas, Texas
Clayton, Mamie .... Ft. Worth, Texas
Cobb, P. C. .... Dallas, Texas
Colvin, Maude .... Ft. Worth, Texas
Cope, Estelle .... Breckenridge, Texas
Cottingham, Ruth .... Dallas, Texas
Crosier, Elizabeth .... Ft. Worth, Texas
Crosier, Lena .... Farmersville, Texas
Cross, E. E. .... Corsicana, Texas
Curba, Addie Mae .... Denton, Texas
Daniels, Mattie R. .... Grandview, Texas
Davidson, Lula B. .... Dallas, Texas
Deen, Carrie .... Dallas, Texas
Dennis, Homer A. .... Dallas, Texas
Denny, Sue .... Dallas, Texas
Dice, Elizabeth .... Dallas, Texas

Drummond, Garrett B.	Ft. Worth, Texas
Dunlap, Rose M.	Petty, Texas
Eason, Leila	Palestine, Texas
Edwards, R. E.	Alvarado, Texas
Elder, Loula	Dallas, Texas
Featherstone, Drury	Alvarado, Texas
Futrell, Carrie	Waco, Texas
Forsythe, L. P.	Corsicana, Texas
Franks, W. D.	Dallas, Texas
Gaither, Caroline	Ft. Worth, Texas
Gatcher, Ellie	Dallas, Texas
Haberger, Mrs. F. J.	Nacogdoches, Texas
Hamilton, Lucy	Dallas, Texas
Harrell, Ruth	Hillsboro, Texas
Harris, Fay	Commerce, Texas
Hartsfield, L. W.	Hillsboro, Texas
Hawes, J. M.	Waco, Texas
Henderson, Mrs. Anna M.	Dallas, Texas
Henry, R. E. L.	Ft. Worth, Texas
Hicks, Pearl	Dallas, Texas
Hightower, Mrs. Louick	Dallas, Texas
Holsapple, Merle	Temple, Texas
Hope, Clarence	Breckenridge, Texas
Hosford, H. M.	Dallas, Texas
Hossell, Kate	Dallas, Texas
Howard, J. F.	San Antonio, Texas
Jackson, A. E.	Houston, Texas
Johnson, Mabel	Ft. Worth, Texas
Johnson, J. C.	Venus, Texas
Johnson, N. H.	Dallas, Texas
Johnson, Pearl	Ft. Worth, Texas
Jones, Deval	Jermyn, Texas
Jones, May	Stephenville, Texas
Kattner, Bruno	Riesel, Texas
Keahey, G. L.	Breckenridge, Texas
Keith, Thos. A.	Ft. Worth, Texas
Kidd, A. E.	Hillsboro, Texas
Killough, Winnie	Dallas, Texas
Kirkman, Ruth	Ft. Worth, Texas
La Grone, Mrs. C. W.	Commerce, Texas

Lawler, Nell A. .... Dallas, Texas
Lewis, Genelia E. .... Ennis, Texas
Lewis, Vera .... Ft. Worth, Texas
Lunsford, D. E. .... Houston, Texas
Marrs, Mary .... Stephenville, Texas
Matthis, F. R. .... Bastrop, Texas
McCarty, Aline .... Barry, Texas
McCormick, Edna .... San Marcos, Texas
McGhee, Mary Beall .... Bonhaw, Texas
McKibben, Dora .... Ft. Worth, Texas
McKinney, Maurine .... Denton, Texas
McLendon, Mina .... Jacksonville, Texas
McMillan, S. W. .... Dallas, Texas
Miller, Harlan .... Dallas, Texas
Minkwitz, Berneta .... Ft. Worth, Texas
Mitchell, Arthur G. .... Sherman, Texas
Mock, P. M. .... Hillsboro, Texas
Moore, Mary Stewart .... Port Arthur, Texas
Moore, O. E. .... Campbell, Texas
Moore, Wilna V. .... Dallas, Texas
Moran, Mrs. Ruby .... Greenville, Texas
Neal, Neva .... Dallas, Texas
Neely, Joe O. .... Denton, Texas
Nelson, W. A. .... Tyler, Texas
Noble, Miss Charlie .... Ft. Worth, Texas
Norman, Mrs. Emma .... Denton, Texas
Norwood, W. H. .... Corsicana, Texas
Obuer, W. L. .... Dallas, Texas
Oebler, James C. .... Venus, Texas
Owen, A. Z. .... Grandview, Texas
Parker, Jane Dawson .... Dallas, Texas
Parks, James E. .... Dallas, Texas
Peeler, Ruth .... Dallas, Texas
Pierce, F. Ben .... Texarkana, Texas
Pierce, John A. .... Corsicana, Texas
Pile, W. A. .... Dallas, Texas
Pinson, R. B. .... Nacogdoches, Texas
Potter, Mary A. .... Racine, Wisconsin
Rasson, Mrs. Virgil .... Longview, Texas
Red, Miss H. L. .... Houston, Texas

Reed, E. D. . . . . Dallas, Texas
Roberts, Gladys . . . . Denton, Texas
Rosenstein, Millie . . . . Ft. Worth, Texas
Rosser, L. E. . . . . Dallas, Texas
Routh, Mrs. W. G. . . . . Hillsboro, Texas
Rorie, George . . . . Dallas, Texas
Scott, Bessie L. Dallas, Texas
Scott, Oliver W. . . . . Hillsboro, Texas
Scott, Ruth . . . . Dallas, Texas
Sellars, Daniel . . . . Ft. Worth, Texas
Sewell, S. M. . . . . San Marcos, Texas
Seymour, Miss Cora A. . . . . Mexico, Texas
Shaw, James Ben . . . . Dallas, Texas
Shelton, Rachel . . . . Alvarado, Texas
Shirley, Pauline . . . . Wichita Falls, Texas
Simmons, Lorita . . . . Ft. Worth, Texas
Smith, Fannie Lewis . . . . Weatherford, Texas
Smith, J. W. . . . . Denton, Texas
Smith, R. N. . . . . Dallas, Texas
Stalcups, J. R. Brownwood, Texas
St. Clare, Florence . . . . Dallas, Texas
Stephens, May . . . . Dallas, Texas
Story, Frank J. Dallas, Texas
Sweeney, Helen . . . . Hillsboro, Texas
Taylor, Florence . . . . Dallas, Texas
Tellery, R. W. . . . . Appleby, Texas
Toonery, Dorothy . . . . Breckenridge, Texas
Turner, Frank . . . . Dallas, Texas
Underwood, P. H. . . . . Galveston, Texas
Vernon, W. C. . . . . San Marcos, Texas
Watson, Mrs. L. W. . . . . Hillsboro, Texas
Watson, S. E. . . . . Alvarado, Texas
Weatherby, D. B. . . . . Corsicana, Texas
Wedgeworth, C. Stanton, Texas
West, J. C. . . . . Ft. Worth, Texas
Whitener, J. A. Ft. Worth, Texas
Williamson, C. S. . . . . Brownwood, Texas
Willard, J. H. . . . . Corsicana, Texas
Winborn, J. E. Houston, Texas
Wolff, E. J. . . . . Ft. Worth, Texas

Modern Plane Geometry. By JOHN CLARK AND ARTHUR S. OTIS. World Book Company. 1927.

John R. Clark and Arthur S. Otis, in their *Modern Plane Geometry,* claim the purpose of Geometry in the High School is "Training the student in methods and habits of thought that result in power to reason and analyze, to discover, and to prove in a logical manner that which has been discovered." No attempt is made to defend Geometry for its direct usefulness in everyday life. It is fundamentally a course in discovery and reasoning. As such it will have utterly lost its value should it degenerate into a memorized reproduction of book-proved theorems.

Without doubt, the strongest feature of the book is the student discovery technique which has been developed. As little as possible is told the pupil, but the stage is set to make discovery natural, and suggestions are made so that the steps in the proof may not be too difficult. The introductory chapter is a skillful effort to bring the student into contact with the materials of geometry naturally and to develop in him the correct attitudes of inquiry, desire for discovery and appreciation of the need for rigorous proof.

The "Starred exercises" are a real incentive to the student to develop independence in thinking. Throughout the book certain of the more difficult exercises are starred. Suggestions for their solution are given at the end of the book—"crutches for those who cannot proceed without them." If a student uses the suggestions, he is entitled to only a "C" on the exercise, if he does it without help, he gets a "B," and after three "B's" in succession the student will receive "A's" until he fails to solve one without the crutches. This device serves not merely as a splendid incentive, but also as a means of self-classification for the student.

"Challenges" occur frequently throughout the book. These aim to encourage the student to "get ahead of the book." Challenge 1. "Read down to the proof in the demonstration of

Proposition 4 on the next page. Cover the proof and see if you can write it out from the plan. If you succeed, enter an "A" after number 1, page 300."

The effect here is clearly to encourage the student to analyze the problem and to anticipate the proof. He must be doing some clear thinking to do this. A year of such exposure must result in developing power to analyze a problem, ability to think clearly, and desire to state the results in concise, unambiguous English.

There is an abundance of interesting and real exercises. These are well organized and classified, but not too well arranged to rob the student of varied applications. Many of the easier theorems have merely suggestions with most of the proof left to the student, others are completely demonstrated for the student not merely for the sake of the demonstration, but also as a specimen of good form.

The general impression that one cannot help getting from studying this book as a reviewer is that the book is written for the student and not for Euclid. It is a real textbook designed to achieve certain results which it sets out consciously to attain. It defines clearly its aim, and then clearly and consistently and constantly drives toward that goal. In reading the book one almost feels his class before him and experiences the thrill of the student as he succeeds in one exercise after another. Some geometry texts seem to be more history books than anything else, in that their aim seems to be to develop an appreciation of the work of Euclid, Pythagoras, and others. This book is all that its name implies, a *Modern* Plane Geometry, with modern ideals of effecting certain changes in the student and with modern technique of setting about deliberately and consciously to attain those changes which in this case are power to reason, to discover and to prove.

H. C. CHRISTOFFERSON

OSHKOSH, WIS.

Introductory Algebra. By ALAN JOHNSON AND ARTHUR BELCHER. New York, F. M. Ambrose Company. Pp. v + 389.

Second Course in Algebra. By ALAN JOHNSON AND ARTHUR BELCHER. New York, F. M. Ambrose Company. Pp. iv + 322.

This two-book series of texts in algebra is written for use in the first year of the standard four-year high school or for use in the ninth grade of the three-year junior high school. The first book is a simplified modern text characterized by clear explanations, desirable eliminations of complex types of exercises, and by adequate problem and drill material. The chapter on numerical trigonometry is particularly satisfactory.

The second book of the series, like the second book of all series of algebras, shows much less change from the textbooks of twenty-five years ago than does the introductory course. Authors do not as yet seem disposed to make any appreciable modification of the second course of algebra. It seems unfortunate that the progress that has resulted in a better selection of subject matter for first-year algebra is not apparent in the second courses.

These books are written by teachers of long experience in the classroom and by persons who have been students of mathematics education. Both books seem to be written with the interest of boys and girls before the authors.